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Euler's Theorem for Homogeneous functions

Introduction

A polynomial in x and y is said to be homogeneous if all its terms are of same degree. For example

$$f(x,y) = x^2 - 2xy + 3y^2$$

is homogeneous. It is easy to generalize the property so that functions (not polynomials) can have this property.

Definition

A function $f(x,y)$ is homogeneous of degree n in a region $D \subseteq \mathbb{R}^2$ iff, for $(x,y) \in D$ and $\lambda > 0$,

$$f(\lambda x, \lambda y) = \lambda^n f(x,y).$$

Note :- A constant function is homogeneous of degree 0. linear functions are homogeneous of degree one.

Example of a non homogeneous function

$$f(x,y) = x^4 + y^4 + 3xy$$

We can see that the power of x in the first term x^4 is 4, the power of y in the second term y^4 is 4, but the sum of the powers of x and y in the third term $3xy$ is $1+1 = 2$.

Hence, $f(x,y)$ is not a homogeneous function

In general, if $f(x,y)$ is a homogeneous function of nth order, then its form will be

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + \text{———} a_{n-1} xy^{n-1} + a_n y^n \quad \text{(i)}$$

We can write $f(x,y)$ in the following alternative way

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + \text{———} a_{n-1} xy^{n-1} + a_n y^n$$

$$= x^n \left[a_0 + \frac{a_1 x^{n-1} y}{x^n} + \text{———} \frac{a_{n-1} xy^{n-1}}{x^n} + \frac{a_n y^n}{x^n} \right]$$

$$= x^n \left[a_0 + a_1 \left(\frac{y}{x}\right) + \text{———} a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n \right]$$

$$= x^n \phi\left(\frac{y}{x}\right),$$ where ϕ is a function of $\frac{y}{x}$.

Euler's theorem

If $u = f(x, y)$ be a continuous function in x and y of degree n, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu.$$

Proof :- Since u is a homogeneous function of the nth degree in x and y we can write u as

$$u = x^n f\left(\frac{y}{x}\right) \text{ ——— (ii)}$$

Differentiating (ii) partially with respect to x, we get

$$\frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right) \cdot -\frac{y}{x^2} + n x^{n-1} f\left(\frac{y}{x}\right)$$

$$= \frac{-x^{n-1} f'\left(\frac{y}{x}\right) \cdot y + n x^n f\left(\frac{y}{x}\right)}{x}$$

$$\Rightarrow x \frac{\partial u}{\partial x} = -x^{n-1} y f'\left(\frac{y}{x}\right) + n x^n f\left(\frac{y}{x}\right)$$

Similarly differentiating (ii) partially with respect to y, we get

$$\frac{\partial u}{\partial y} = x^n . f'\left(\frac{y}{x}\right) . \frac{1}{x}$$

$$= x^{n-1} f'\left(\frac{y}{x}\right)$$

Hence $$y \frac{\partial u}{\partial y} = x^{n-1} y f'\left(\frac{y}{x}\right)$$

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So,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \cancel{-x^{n-1}y f'\left(\frac{y}{x}\right)}+nx^n f\left(\frac{y}{x}\right)$$

$$\cancel{+x^{n-1}y f'\left(\frac{y}{x}\right)}$$

$$= nu \qquad \boxed{\text{Since } u = x^n f\left(\frac{y}{x}\right).}$$

Examples

1. If $\sin u = \dfrac{x^3+y^3}{x-y}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\tan u.$$

Given :- $\sin u = \dfrac{x^3+y^3}{x-y}$

$$= \frac{x^3\left(1+\frac{y^3}{x^3}\right)}{x\left(1-\frac{y}{x}\right)}$$

$$= x^2\phi\left(\frac{y}{x}\right)$$

$\therefore$ $\sin u$ is a homogeneous function of order 2 in x and y.

Let $v = \sin u$

According to Euler's theorem

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y} = 2v$$

$$\Rightarrow x\cos u.\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = 2\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u} = 2\tan u.$$

Proved

2. If $u = \tan^{-1} \dfrac{x^2+y^2}{x+y}$ then show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \sin 2u.$$

Since $u = \tan^{-1} \dfrac{x^2+y^2}{x+y}$

$$\Rightarrow \tan u = \frac{x^2+y^2}{x+y}$$

$$\Rightarrow \tan u = \frac{x^2\left(1+\frac{y^2}{x^2}\right)}{x\left(1+\frac{y}{x}\right)} = x\,\phi\left(\frac{y}{x}\right)$$

Hence, $\tan u$ is a homogeneous function of order 1.

Let $v = \tan u$.

By Euler's theorem

$$x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} = v$$

$$\Rightarrow x \sec^2 u \cdot \frac{\partial u}{\partial x} + y \sec^2 u \cdot \frac{\partial u}{\partial y} = \tan u.$$

$$\Rightarrow x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\tan u}{\sec^2 u}$$

$$= \frac{\sin u \cdot \cos u \cdot \cancel{\cos u}}{\cancel{\cos u}}$$

$$= \frac{1}{2} \sin 2u.$$

Proved. [$\because \sin 2u = 2 \sin u \cdot \cos u$

All students will finish the exercise of 3(B).

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