Sub-GROUP

If $H \subseteq G$ and G is a group with $\cdot$
And ~~then~~ if H is also a group with $\cdot$
Then H is called a sub-group of G

As an example we can see that the set Q of Rationals is a group with $+$. Also I, the set of all integers is also a group with $+$ and also, $I \subseteq Q$ so, I is a subgroup of Q.

Similarily the set R_1 of all non-zero real numbers forms a group with $(\cdot)$ Further Z_1 the set of all non-zero Rational Numbers is $\subseteq R_1$ also Z_1 is a group with $(\cdot)$ Thus Z_1 is a Subgroup of R_1.

<u>Theorem</u> A non-empty subset ~~of~~ H of a group G is a subgroup of G iff for $a, b \in H$, $ab^{-1} \in H$

iff = if and only if

Proof: ~~Since G is a~~ Let $H \subseteq G$ and and H is a sub group of G. So, H is a group in itself so,

$$a, b \in H \Rightarrow ab^{-1} \in H$$

Hence condition is necessary.

For Sufficient part.

Let H ~~is~~ $\subseteq G$ and $a, b, \in H$ we have ~~How to prove~~ $ab^{-1} \in H$.

Further $e, a^{-1} \in H \Rightarrow e a^{-1} \in H$
i.e. $a \in H \Rightarrow a^{-1} \in H$
So inverse element exists for each element of G. Also $a, b \in H \Rightarrow a, b^{-1} \in H \Rightarrow ab \in H$
So the operation is binary in H
Further operation is associative in G
So it must be associative in H.
Thus H is a group by itself
and so H is a subgroup of G

Theorem If $\{H_\alpha\}_{\alpha \in I}$ is a family of subgroups of G. Then $H = \bigcap_\alpha (H_\alpha)$ is also a subgroup of G.
Alternatively we can say that Intersection of Subgroups is also a subgroup.
OR The set of Subgroups is closed under Intersection.

Proof →
(i) The identity element 'e' is in each H_α
So, $e \in \bigcap_\alpha H_\alpha$ i.e. $e \in H$.
(ii) If ~~a∈H~~ $a \in H$ then $a \in$ each H_α
and $a \in H_\alpha \Rightarrow a^{-1} \in H_\alpha$ so, $a^{-1} \in H$
Thus $a \in H \Rightarrow a^{-1} \in H$
(iii) Also $a, b \in H \Rightarrow a, b \in$ each H_α
$\Rightarrow ab^{-1} \in$ each H
$\Rightarrow ab^{-1} \in H$.

Thus by (i) (ii) & (iii), we have
H is a Sub group of G.

Theorem :- If H_1 and H_2 are two subgroups of G then $H_1 \cup H_2$ need not be a subgroup of G

Proof We prove this by an example
let $H_1 = \{0, \pm 2, \pm 4, \pm 6 \text{ --- }\}$
i.e H_1 is set of all even integers

and $H_2 = \{0, \pm 3, \pm 6, \pm 9 \text{ --- }\}$
i.e. H_2 is set of all multiples (integral) of 3.

Now H_1 is a Group ~~of~~ with addition because $+$ is Binary (Addition of even integers is even)
$+$ is Associative
$0 \in H_1$ and for $p \in H_1$ $-p \in H_1$
So $(H_1, +)$ is a group.

Next H_2 is also a group with $+$ because again (i) adding any two multiples of 3 will give a multiple of 3
~~$0 \in H_2$~~ So $+$ is Bi...

for any $x \in H_2$ $-x \in H_2$

Thus H_1 & H_2 are both groups with +

However $H_1 \cup H_2$ is not a group
because $2 \in H_1$, $3 \in H_2$
But ~~So~~ $2+3=5 \notin H_1 \cup H_2$
Thus the operation is not Binary
So Union of two Sub groups need not be a Sub-Group.